Final Assignment 2

Essential and Density Topologies on Continuous Domains

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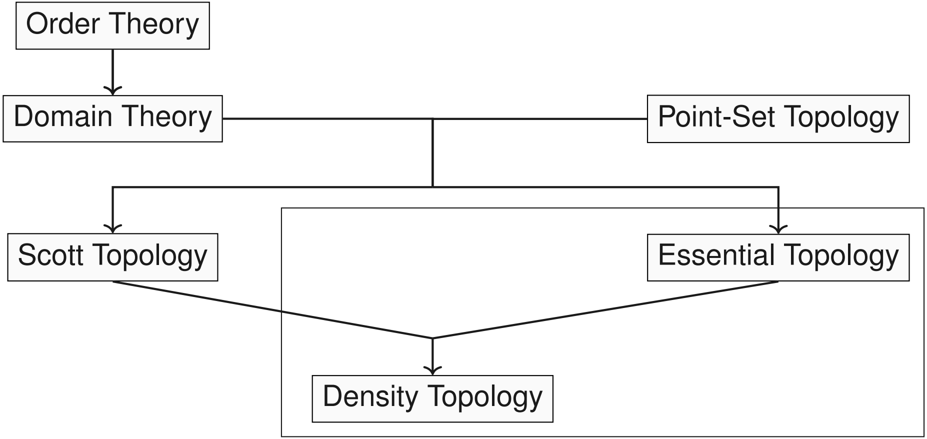
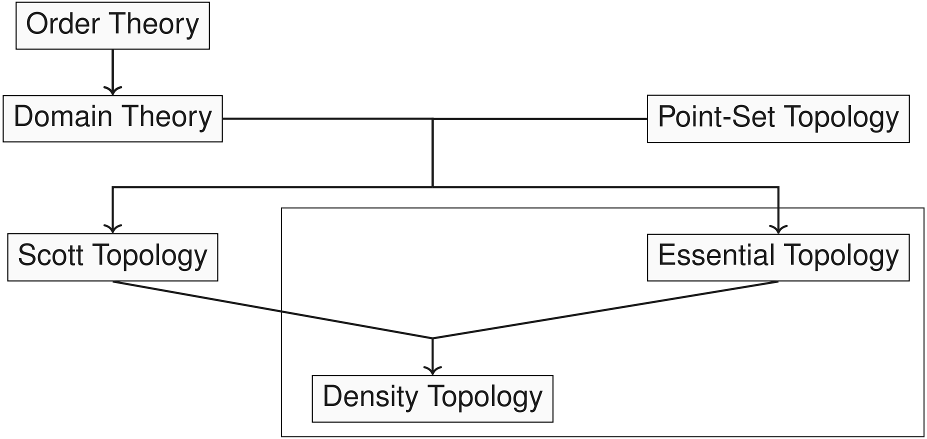
Concept Mapping



Given by [Rusu and Ciobanu (2016)], main focus of thesis



Given by [Rusu and Ciobanu (2016)], main focus of thesis

Why? We will construct a topology such that **bases for continuous domain coincide with dense sets.**

Definition 1 (Partially ordered set)

Let X be a non-empty set. A relation $\leq$ on the set X is a **partial order** if $\leq$ is reflective, antisymmetric, and transitive. The pair $(X, \leq)$ is referred to as a partially ordered set (poset).

Definition 1 (Partially ordered set)

Let X be a non-empty set. A relation $\leq$ on the set X is a **partial order** if $\leq$ is reflective, antisymmetric, and transitive. The pair $(X, \leq)$ is referred to as a partially ordered set (poset).

Definition 2 (Directed set)

The non-empty set $D \subseteq X$ is **directed** provided that every pair of elements in D has an upper bound in D.

Definition 3 (Directed-complete posets (dcpo))

The poset $(X, \leq)$ is a **directed-complete poset (dcpo)** if every directed subset of X admits a supremum.

This thesis will focus on directed-complete posets (dcpos).

Definition 4 (Topological Spaces)

Let X be a non-empty set. The collection $\tau \subseteq \mathcal{P}(X)$ is a topology if it satisfies the following axioms:

(T1) $\varnothing, X \in \tau$;

(T2) For all $A, B \in \tau$, $A \cap B \in \tau$;

(T3) For all subcollections $\mathcal{C} \subseteq \tau$, $\bigcup \mathcal{C} \in \tau$.

The ordered pair (X, τ) is a **topological space** and a set $A \in \tau$ is referred to as an **open** set.

Definition 5

The set $A \subseteq X$ is a **closed** set if $X \backslash A$ is open.

We refer to a set that is both open and closed as a **clopen** set.

Definition 6

For every $x \in X$, define $\uparrow x = \{y \in X \mid x \leq y\}$. Furthermore, for any given subset $A \subseteq X$, let $\uparrow A = \bigcup_{x \in A} \uparrow x$. Analogously, let $\downarrow x = \{y \in X \mid y \leq x\}$ and $\downarrow A = \bigcup_{x \in A} \downarrow x$.

A set A is an **upper (lower) set** provided that $A = \uparrow A$ ($A = \downarrow A$).

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Proposition 7

*The collection $\tau = \{A \mid A \subseteq X, A = \uparrow A\}$ is a topology and referred to as the **Alexandroff topology**.*

Let $(X, \leq)$ be a dcpo.

Definition 8

For any x or $y \in X$, we say that x is an **essential part** of y or x is **way-below** y, denoted by $x \ll y$, if for every directed set $D \subseteq X$ such that $y \leq \sup D$, there exists $d \in D$ such that $x \leq d$.

Essential/Way-Below Relation

Let $(X, \leq)$ be a dcpo.

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Definition 9

For any $x \in X$, we say that x is **compact** if x is an essential part of x itself. The set of all compact elements of X is denoted by $K(X)$.

Let $(X, \leq)$ be a dcpo.

Definition 10

For every $x \in X$, define $\twoheaduparrow x = \{y \in X \mid x \ll y\}$. Furthermore, for any given subset $A \subseteq X$, let $\twoheaduparrow A = \bigcup_{x \in A} \twoheaduparrow x$. Analogously, let $\twoheaddownarrow x = \{y \in X \mid y \ll x\}$ and $\twoheaddownarrow A = \bigcup_{x \in A} \twoheaddownarrow x$.

Essential Topology

Let $(X, \leq)$ be a dcpo.

Definition 10

For every $x \in X$, define $\Uparrow x = \{y \in X \mid x \ll y\}$. Furthermore, for any given subset $A \subseteq X$, let $\Uparrow A = \bigcup_{x \in A} \Uparrow x$. Analogously, let $\Downarrow x = \{y \in X \mid y \ll x\}$ and $\Downarrow A = \bigcup_{x \in A} \Downarrow x$.

Definition 11 ([Rusu and Ciobanu (2016)])

A set $A \subseteq X$ is an **e-open** set if $\Downarrow A \subseteq A$; it is an **e-closed** set if $\Uparrow A \subseteq A$; furthermore, when A is concurrently e-open and e-closed, it is referred to as an e-clopen set.

Proposition 12

*The collection of e-open sets on a dcpo $(X, \leq)$ a topology. This topology is referred to as the **essential topology** or **e-topology**, denoted by τ_e.*

Let $(X, \leq)$ be a dcpo.

Definition 13

A set $O \subseteq X$ is **Scott-open** if it is an upper set and for every directed set D with $\sup D \in O$, the set $D \cap O$ is non-empty.

The complement of a Scott-open set is a **Scott-closed** set.

Proposition 14

The collection of Scott-open set, σ_X, is a topology on X, and referred to as the ***Scott topology****.*

The definition of the density topology is provided by [Rusu and Ciobanu (2016)].

Proposition 15 ([Rusu and Ciobanu (2016)])

Let $(X, \leq)$ *be a dcpo. The collection* $\mathcal{D} = \{D \cap G \mid D \in \sigma_X, G \in \tau_e\}$ *generates a topology on* X*. That is,*

$$\rho_X = \left\{\bigcup \mathcal{D}' \mid \mathcal{D}' \subseteq \mathcal{D}\right\}.$$

The topology ρ_X *is referred to as the* ***density topology****.*

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Proposition 15 ([Rusu and Ciobanu (2016)])

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$$\rho_X = \left\{ \bigcup \mathcal{D}' \mid \mathcal{D}' \subseteq \mathcal{D} \right\}.$$

The topology ρ_X is referred to as the ***density topology****.*

Proposition 16

The density topology is the smallest common refinement of the essential topology and the Scott topology.

Domains and Basis

Definition 17

The dcpo $(X, \leq)$ is a **continuous domain** (or a domain) provided that for any $x \in X$, the set $\downarrow x$ is directed and $x = \sup(\downarrow x)$.

Definition 18

The dcpo $(X, \leq)$ is an **algebraic domain** provided that for any $x \in X$, the set $\downarrow x \cap K(X)$ is directed and $x = \sup(\downarrow x \cap K(X))$.

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The dcpo $(X, \leq)$ is an **algebraic domain** provided that for any $x \in X$, the set $\downarrow x \cap K(X)$ is directed and $x = \sup(\downarrow x \cap K(X))$.

Definition 19

A set $B \subseteq X$ is a **basis for** X provided that for every $x \in X$, the set $B_x := B \cap \downarrow x$ contains a directed set D_x such that x is the supremum of D_x.

One can prove that a dcpo $(X, \leq)$ is a continuous domain if and only if it has a basis.

Example 20 (Extended Real Line)

Consider the dcpo $(\mathbb{R}_\infty, \leq)$ *where* $\leq$ *is the usual ordering of* $\mathbb{R}_\infty = (-\infty, \infty]$*. One can assert that* $x \ll y$ *iff* $x < y$ *for any* x*,* $y \in \mathbb{R}_\infty$*. The set of rational numbers* $\mathbb{Q}$ *is a basis for* $\mathbb{R}_\infty$*.*

Example 20 (Extended Real Line)

Consider the dcpo $(\mathbb{R}_\infty, \leq)$ where $\leq$ is the usual ordering of $\mathbb{R}_\infty = (-\infty, \infty]$. One can assert that $x \ll y$ iff $x < y$ for any x, $y \in \mathbb{R}_\infty$. The set of rational numbers $\mathbb{Q}$ is a basis for $\mathbb{R}_\infty$.

For any set X, $\mathcal{P}_{\text{fin}}(X) := \{A \in \mathcal{P}(X) \mid |A| < \infty\}$

Example 21 (Power Set)

Consider the dcpo $(\mathcal{P}(X), \subseteq)$ where X is a set. One can assert that $A \ll B$ iff $A \in \mathcal{P}_{\text{fin}}(B)$ for any A, $B \in \mathcal{P}(X)$. The set $\mathcal{P}_{\text{fin}}(X)$ is a basis for $\mathcal{P}(X)$. Furthermore, $\mathcal{P}_{\text{fin}}(X) = K(\mathcal{P}(X))$.

Let $(X, \leq)$ be a dcpo.

Proposition 22

For every $x \in X$, there exists the smallest e-open set that contains x; that is, $V_x := \{x\} \cup \downarrow\!\!\!\!\downarrow x$.

Proposition 23

The collection $\{\{x\} \cup \downarrow\!\!\!\!\downarrow x \mid x \in X\}$ is a basis for the topology τ_e.

Example 24 (Extended Real Line)

Consider the continuous domain $(\mathbb{R}_\infty, \leq)$ *where* $\leq$ *is the usual ordering of* $\mathbb{R}_\infty = (-\infty, \infty]$. *For any* $x \in \mathbb{R}_\infty$, $\downarrow x = (-\infty, x)$, *we have* $V_x = (-\infty, x]$. *Thus, every e-open set can be expressed in the form of* $(-\infty, a]$ *or* $(-\infty, a)$ *where* $a \in \mathbb{R}_\infty$.

Example 24 (Extended Real Line)

Consider the continuous domain $(\mathbb{R}_\infty, \leq)$ *where* $\leq$ *is the usual ordering of* $\mathbb{R}_\infty = (-\infty, \infty]$. *For any* $x \in \mathbb{R}_\infty$, $\downarrow x = (-\infty, x)$, *we have* $V_x = (-\infty, x]$. *Thus, every e-open set can be expressed in the form of* $(-\infty, a]$ *or* $(-\infty, a)$ *where* $a \in \mathbb{R}_\infty$.

Example 25 (Power Set)

Consider the algebraic domain $(\mathcal{P}(X), \subseteq)$ *where* X *is a set. The set* $\mathcal{O} \subseteq \mathcal{P}(X)$ *is e-open iff*

$$\bigcup_{A \in \mathcal{O}} \mathcal{P}_{\text{fin}}(A) = \bigcup_{A \in \mathcal{O}} \downarrow A = \downarrow \mathcal{O} \subseteq O$$

which is equivalent to say that for every $A \in \mathcal{O}$, *it holds that* $\mathcal{P}_{\text{fin}}(A) \subseteq \mathcal{O}$.

Basis for Density Topology

Recall that the collection $\{\{x\} \cup {\downarrow} x \mid x \in X\}$ is a basis for the e-topology on a dcpo $(X, \leq)$.

Proposition 26

Let $(X, \leq)$ be a continuous domain, then the collection $\{{\uparrow} x \mid x \in X\}$ is a basis for the Scott topology.

Recall that the collection $\{\{x\} \cup \downarrow x \mid x \in X\}$ is a basis for the e-topology on a dcpo $(X, \leq)$.

Proposition 26

Let $(X, \leq)$ be a continuous domain, then the collection $\{\uparrow x \mid x \in X\}$ is a basis for the Scott topology.

Proposition 27

Let $(X, \leq)$ be a continuous domain. The collection $\{\uparrow x \cap (\{y\} \cup \downarrow y) \mid x, y \in X\}$ is a basis for the density topology.

Example 28 (Extended Real Line)

Consider the continuous domain $(\mathbb{R}_\infty, \leq)$. *The collection*

$$\{(x, \infty] \mid x \in \mathbb{R}_\infty\}$$

is a basis for the Scott topology $\sigma_{\mathbb{R}_\infty}$. *On the other hand, the collection*

$$\{(-\infty, x] \mid x \in \mathbb{R}_\infty\}$$

is a basis for the e-topology on $\mathbb{R}_\infty$. *Hence, by Proposition 27 the collection*

$$\{(x, \infty] \cap (-\infty, y] \mid x, y \in \mathbb{R}_\infty\} = \{(x, y] \mid x, y \in \mathbb{R}_\infty\}.$$

is a basis for the density topology $\rho_{\mathbb{R}_\infty}$.

Proposition 29

Let $(X, \leq)$ *be a continuous domain with* $B \subseteq X$ *being the basis for* X*, then the set* B *is e-dense in* X*.*

Proposition 29

Let $(X, \leq)$ be a continuous domain with $B \subseteq X$ being the basis for X, then the set B is e-dense in X.

We can simplify the characterization of an e-dense set as follows.

Proposition 30

Let $(X, \leq)$ be a dcpo where $\Uparrow X = X$ and $B \subseteq X$. The following assertions are equivalent:

(1) *B is e-dense;*

(2) *$\Uparrow B = X$;*

(3) *$X \backslash B \subseteq \Uparrow B$.*

Basis Through Essential Topology

Theorem 31

Let $(X, \leq)$ be a continuous domain and $B \subseteq X$. The following assertions are equivalent:

(1) B *is a basis for* X*;*

(2) $\Uparrow(\Uparrow x \cap B) = \Uparrow x$ *for all* $x \in X$*;*

(3) $\Uparrow(\Uparrow A \cap B) = \Uparrow A$ *for all* $A \subseteq X$*;*

(4) $\Uparrow(F \cap B) = \Uparrow F$ *for all e-closed set* F*;*

(5) $\mathrm{cl}_e(\Uparrow A \cap B) = \Uparrow A$ *for all* $A \subseteq X$*;*

(6) *for every* $D \in \sigma_X$ *and* $G \in \tau_e$ *such that* $D \cap G \neq \varnothing$*, we have* $D \cap G \cap B \neq \varnothing$*.*

An e-Dense Set is not Always a Basis for a Continuous Domain

Recall:

- Theorem 31((1)⇔(2)) establishes that the set B serves as a basis for X if we can employ B to interpolate to elements.
- Proposition 29 establishes that a basis for X is an e-dense set.

An e-Dense Set is not Always a Basis for a Continuous Domain

Recall:

- Theorem 31((1)⇔(2)) establishes that the set B serves as a basis for X if we can employ B to interpolate to elements.
- Proposition 29 establishes that a basis for X is an e-dense set.

However, the converse: e-dense set is a basis, does not always hold true.

Example 32 (Extended Real Line)

Consider the continuous domain $(\mathbb{R}_\infty, \leq)$. *Since* $\Uparrow\mathbb{Z} = \mathbb{R}_\infty$, *the set* $\mathbb{Z}$ *is e-dense by Proposition 30((1)⇔(2)). However,* $\mathbb{Z}$ *is not a basis for* $\mathbb{R}_\infty$ *since by taking* 0 *and* $1 \in \mathbb{R}_\infty$, *we cannot find any element* $b \in \mathbb{Z}$ *such that* $0 \ll b \ll 1$. *From Theorem 31((1)⇔(2)), we deduce that* $\mathbb{Z}$ *is not a basis for* $\mathbb{R}_\infty$.

Corollary 33

Let $(X, \leq)$ be a continuous domain. The non-empty set B is a basis for X if and only if B is e-dense in each non-empty Scott-open set; that is, $\mathrm{cl}_e(A \cap B) = A$ for all $A \in \sigma_X$.

Proof. Theorem 31((1)⇔(5))

Corollary 33

Let $(X, \leq)$ be a continuous domain. The non-empty set B is a basis for X if and only if B is e-dense in each non-empty Scott-open set; that is, $\mathrm{cl}_e(A \cap B) = A$ for all $A \in \sigma_X$.

Proof. Theorem 31((1)⇔(5))

Theorem 34

Let $(X, \leq)$ be a continuous domain. The continuous domain $(X, \leq)$ is algebraic if and only if $\mathrm{cl}_e(F \cap K(X)) = \uparrow F$ for every e-closed set F.

Proof. Theorem 31((1)⇔(4)) and 31((1)⇔(5))

Basis for Continuous Domain iff ρ_X-dense

We shall assess the main inquiries of this thesis; that is, finding a topology such that bases for a continuous domain coincides with dense sets in such topology.

Theorem 35

If $(X, \leq)$ is a continuous domain, the non-empty set $B \subseteq X$ is a basis for X if and only if B is ρ_X-dense in X.

Proof. Theorem 31((1)⇔(6))

We shall assess the main inquiries of this thesis; that is, finding a topology such that bases for a continuous domain coincides with dense sets in such topology.

Theorem 35

If $(X, \leq)$ is a continuous domain, the non-empty set $B \subseteq X$ is a basis for X if and only if B is ρ_X-dense in X.

Proof. Theorem 31((1)⇔(6))

Definition 36

The collection of all ρ_X-dense subsets of X is denoted by $\mathcal{B}_X$.

Theorem 37

Let $(X, \leq)$ be a continuous domain.

(i) $x \in K(X)$ *if and only if* $\{x\} \in \rho_X$*;*

(ii) $K(X) \in \rho_X$*;*

(iii) *If* $K(X) \neq \varnothing$*, then* $K(X) \backslash \{x\} \notin \mathcal{B}_X$ *for all* $x \in K(X)$*;*

(iv) *If* $B \in \mathcal{B}_X$ *and* $x \notin K(X)$*, then* $x \in \mathrm{cl}_{\rho_X}(B \backslash \{x\})$*;*

(v) *If* $B \in \mathcal{B}_X$ *and* $x \notin K(X)$*, then* $B \backslash \{x\} \in \mathcal{B}_X$*;*

(vi) $K(X) = \bigcap \mathcal{B}_X$*;*

(vii) $K(X) \subseteq \mathrm{int}_{\rho_X} B$ *for all* $B \in \mathcal{B}_X$*.*

A set A in a topological space (X, τ) is nowhere dense if $\mathrm{int}_\tau(\mathrm{cl}_\tau A) = \varnothing$.

Theorem 38

Let $(X, \leq)$ *be a continuous domain. The following assertions are equivalent:*

(1) $(X, \leq)$ *is an algebraic domain;*

(2) $\bigcap \mathcal{B}_X \in \mathcal{B}_X$;

(3) $(\mathcal{B}_X, \supseteq)$ *is a ccpo;*

(4) $(\mathcal{B}_X, \supseteq)$ *is a dcpo;*

(5) $\bigcap (\mathcal{B}_X \cap \rho_X) \in \mathcal{B}_X$;

(6) *union of any collection of non-empty* ρ_X*-nowhere dense sets is not* ρ_X*-open.*

Summary: e-topology

1. The essential topology is defined with a motivation akin to that of the Alexandroff topology.
2. Essential topology is an intermediate topology to answer our main inquiries: construction of a topology such that bases for a continuous domain coincide with dense sets in such topology.
3. We are able to gain new characterization of a bases and algebraic domains through essential topology (See Theorem 31).
4. However, essential topology is not sufficient to assert bases coincide with dense sets (See Example 32).

Summary: Density Topology

1. The density topology is defined as the smallest refinement of essential topology and Scott topologies.
2. Within the context of density topology, bases for a continuous domain concide with ρ_X-dense sets (See Theorem 35).
3. Using density topology, we are able to provide a topological view for the properties of the bases and compact elements (See Theorem 37).
4. Additionally, we are able to characterize an algebraic domain through density topology (See Theorem 38).

Future Inquiries

1. Future research may build upon the findings of this study by investigating additional properties such as connectivity, local compactness, and the convergence of the specified topologies.

2. Given that both examined topologies are associated with the Scott topology, it appears plausible to investigate the relationship between continuous functions defined on these novel topologies and Scott-continuous functions, as elaborated by [Scott (1972)].

3. The generalization of the results from this study could be explored; for instance, by analyzing the essential and density topologies of SI-topology through the lens of the I-way-below relation, as presented by [Andradi (2018)].

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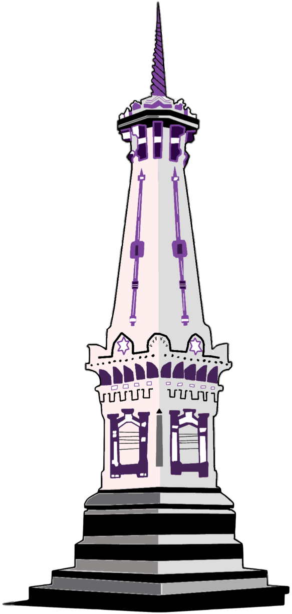

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