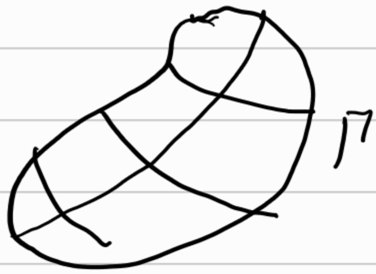
Calculus of Variations (Lecture 23) 25/05/2026

Douglas - Rado Solution to the Plateau Problem

Plateau Problem:

Given a curve $\Gamma \subseteq \mathbb{R}^3$, find a surface Σ with $\partial\Sigma = \Gamma$ and such that Area(Σ) is the minimal one.



Douglas - Rado Setting:

- Γ is a smooth Jordan curve, i.e. $\Gamma = \gamma(S^1)$, $\gamma : S^1 \to \mathbb{R}^3$ smooth embedding: γ is bijective, $|\dot\gamma| \neq 0$
- A smooth surface is a map $\Phi : \overline{D} \subseteq \mathbb{R}^2 (= \mathbb{C}) \to \mathbb{R}^3$ smooth: (sometimes)

  1) $\Phi|_{\partial D = S}$ is an injective parametrization of Γ
  2) $\partial_x \Phi \wedge \partial_y \Phi \neq 0$
- We define

$$\text{Area}(\Sigma) = \text{Area}(\Phi) = \int_D |\partial_x \Phi \wedge \partial_y \Phi|$$

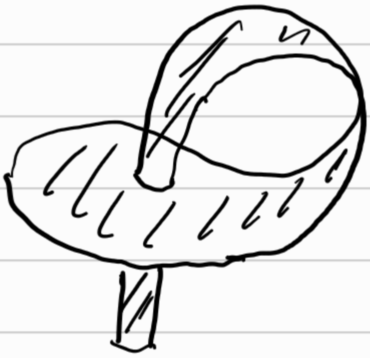
$$= \int_D \sqrt{\sum_{i=1}^{3} (\det M_i)^2}$$

where M_i is the i^{th} 2×2 minor of $D\Phi$.

Remark.

1) We are prescribing the topology of the image (disk-like), so no handles

2) There might be self intersection



$\leftarrow$ In this case we can decrease the area by increasing the genus (Pictures on the website of E. Podini (Pisa))

3) Area is invariant by reparametrization, i.e., if $\Psi : \overline{D} \to \overline{D}$ is a diffeomorphism and Φ is admissible, then $\Phi \circ \Psi$ is admissible and

$$\text{Area}(\Phi \circ \Psi) = \text{Area}(\Phi)$$

We want to apply direct methods

Theorem. Let $\{\Phi_k\} \subseteq W^{1,2}(D;\mathbb{R}^3)$ such that $\Phi_k \xrightarrow{W^{1,2}} \Phi \in W^{1,2}(D;\mathbb{R}^3)$. Then, $\text{Area}(\Phi) \le \liminf_{k\to\infty} \text{Area}(\Phi_k)$

Remark. $W^{1,2}$ is "natural": $\text{Area}(\Phi) = \int |\partial_x \Phi \wedge \partial_y \Phi| \le \int |\partial_x \Phi| \cdot |\partial_y \Phi|$

$$\le \frac{1}{2}\int |\partial_x \Phi|^2 + |\partial_y \Phi|^2$$

Every $\Phi \in W^{1,2}(D;\mathbb{R}^3)$ has finite area.

Proof (Sketch): Area is polyconvex

1) As what we did in Lecture 12:

$\forall \varphi \in C_c^\infty(D;\mathbb{R}^3)$: $\displaystyle\int \varphi\,(\partial_x \Phi_k^i\, \partial_y \Phi_k^j - \partial_x \Phi_k^j\, \partial_y \Phi_k^i)$

equals $\partial_x(\Phi_k^i \partial_y \Phi_k^j) - \partial_y(\Phi_k^i \partial_x \Phi_k^j)$

$\Big\downarrow$ $k\to\infty$ by weak conv

$$\int \varphi\,(\partial_x \Phi^i\, \partial_y \Phi^j - \partial_x \Phi^j\, \partial_y \Phi^i)$$

2) $\forall \varphi = (\varphi^1, \varphi^2, \varphi^3) \in C_c^\infty(D;\mathbb{R}^3)$, $|\varphi| \le 1$

$$\text{Area}(\Phi) = \int |\partial_x \Phi \wedge \partial_y \Phi| \ge \int \varphi \cdot (\partial_x \Phi \wedge \partial_y \Phi)$$

$$= \int \varphi^1(\partial_x \Phi^2 \partial_y \Phi^3 - \partial_x \Phi^3 \partial_y \Phi^2) + \dots$$

and so

$$\text{Area}(\Phi) = \sup_{\substack{|\varphi| \le 1 \\ \varphi \in C_c^\infty(D; \mathbb{R}^3)}} \int \varphi(\partial_x \Phi \wedge \partial_y \Phi).$$

By (1), each of the maps $\Phi \longmapsto \int \varphi(\partial_x \Phi \wedge \partial_y \Phi)$ is continuous.

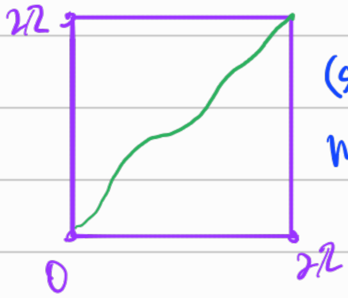
Combining these two ideas, we have L.S.C.

Definition. Our admissible class is going to be

$$\tilde{\mathcal{C}}(\Gamma) = \left\{ \Phi \in W^{1,2}(D; \mathbb{R}^3) \cap C^0(\bar{D}; \mathbb{R}^3) : \Phi|_{\partial D} \text{ is a monotone parametrization of } \Gamma,\ \Phi(S^1) = \Gamma \right\}$$

where $\Phi : \partial D \simeq S^1 \longrightarrow \Gamma$ is a (weakly) monotone parametrization if $\gamma^{-1} \circ \Phi : S^1 := \mathbb{R}/_{[0,2\pi]} \longrightarrow S^1 \simeq \mathbb{R}/_{[0,2\pi]}$ is (weakly monotone:

$\Gamma = \gamma(S^1)$, Φ monotone if



(strictly) monotone: $\theta_1 < \theta_2 \Rightarrow \gamma^{-1} \circ \Phi(\theta_1) < \gamma^{-1} \circ \Phi(\theta_2)$

weakly monotone: $\theta_1 < \theta_2 \Rightarrow \gamma^{-1} \circ \Phi(\theta_1) \le \gamma^{-1} \circ \Phi(\theta_2)$

Remark/Exercise. The uniform closure of (strictly) monotone maps from $[0,2\pi]$ into $[0,2\pi]$ are the weakly monotone maps.

Remark. Area is not coercive on $\mathcal{C}(\Gamma)$. There exists a sequence $\varphi_k : \bar{D} \longrightarrow \bar{D}$ diffeo such that $\varphi_k|_{\partial D} = id$ and such that $\varphi_k(x) \to$ constant $\forall x \in D = \{(x,y) : \sqrt{x^2+y^2} < 1\}$.

↳ **Example.** Consider $f_k : [0,1] \longrightarrow [0,1]$ monotone such that $f_k(0), f_k(1) = 1$, but $f_k \longrightarrow 0$ for $x \in [0,1)$.

Then, we define $\varphi_k(\rho, \theta) = f_k(\rho) \cdot (\cos\theta, \sin\theta)$.

Pick any $\Phi \in \tilde{\mathcal{C}}(\Gamma)$, $\Phi_k = \Phi \circ \varphi_k \in \tilde{\mathcal{C}}(\Gamma)$ where $\text{Area}(\Phi_k) = \text{Area}(\Phi)$, but $\underbrace{\Phi_k \xrightarrow{a.e.} \overbrace{\text{constant map} \notin \tilde{\mathcal{C}}(\Gamma)}^{\text{not a (weakly) monotone parametrization}}}_{\text{if it converges}}$.

A Detour on Geodesics

Let $\Omega \subseteq \mathbb{R}^d$; $g : \Omega \to \text{Sym}_d(\mathbb{R})$ a Riemannian metric and
$y \mapsto g_{ij}(y)$ s.t. $\Lambda(\xi)^2 \geq g_{ij}\xi^i\xi^j \geq \lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^d$

For $\gamma : [0,1] \longrightarrow \Omega$, we define
$$L(\gamma) = \int_0^1 \sqrt{g_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t)}\, dt = \int_0^1 |\dot{\gamma}|_{g(\gamma(t))}\, dt$$

and consider the variational problem for $P, Q \in \Omega$:

$$(\mathcal{P}) \quad \min\{L(\gamma) : \gamma(0) = P,\ \gamma(1) = Q\}$$

Note that for $\varphi : [0,1] \longrightarrow [0,1]$ smooth monotone map
$$L(\gamma) = L(\gamma \circ \varphi).$$

Remark. Length is l.s.c. on $W^{1,p}$ $\forall p \in [1, +\infty)$, but length is not coercive on $W^{1,p}$ (because of invariance by reparametrization as in Area case).

Definition. For $\gamma \in W^{1,2}([0,1]; \Omega)$, $E(\gamma) = \int_0^1 |\dot{\gamma}|^2 = \int_0^1 g_{ij}(\gamma(t))\dot{\gamma}^i\dot{\gamma}^j\, dt$

Remark.

1) By C-S ineq:
$$L(\gamma) \leq \sqrt{E(\gamma)}$$
where equality iff $|\dot{\gamma}| = \text{constant} = L(\gamma)$

2) If we reparametrize γ by a (multiple of) arc-length, then
$$L(\gamma) = \sqrt{E(\tilde{\gamma})}$$
$\uparrow$ arclength reparametrization of γ.

So now we can consider the minimization problem

(P') $\quad \min \{ E(\gamma) : \gamma(0) = P, \gamma(1) = Q \}$

Then $\inf L(\gamma) \leq \sqrt{\inf E(\gamma)}$ $\quad [(P) \leq \sqrt{(P')}]$

but actually, we can pick any γ and let $\tilde{\gamma}$ be the arclength reparametrization, then

$$\sqrt{\inf E(\gamma)} \leq \sqrt{E(\tilde{\gamma})} = L(\tilde{\gamma}) = L(\gamma)$$

and we conclude $\inf L(\gamma) \geq \sqrt{\inf E(\gamma)}$

Minimizing the energy is the same as minimizing the length and this also should unify that the minimizer of the energy should have a constant speed reparametrization.

Actually, the fact that the minimizer of the energy has constant speed follows from inner variations:

Lemma. Let γ be a minimizer of $E(\gamma)$, then $|\dot{\gamma}| = \text{constant}$.

Proof. (DuBois-Reimond Condition).

Let γ be the minimizer, $\lambda \in C_c^1([0,1]; \mathbb{R})$ and $\varphi_\varepsilon(s) = s + \varepsilon\lambda(s)$.

Note that $\varphi_\varepsilon : [0,1] \to [0,1]$, $\varphi_\varepsilon' \geq 1 - C_\varepsilon > 0$ $\quad (\varepsilon \ll 1)$,

and $\varphi_\varepsilon(0) = 0$, $\varphi_\varepsilon(1) = 1$ $\quad$ (φ_ε is a Diffeo)

By setting $\gamma_\varepsilon = \gamma \circ \varphi_\varepsilon^{-1}$:

$$E(\gamma_\varepsilon) = \int_0^1 g_{ij}(\gamma(\varphi_\varepsilon^{-1}(s)))\, \dot{\gamma}^i(\varphi_\varepsilon^{-1}(s))\, \dot{\gamma}^j(\varphi_\varepsilon(s)) \cdot (\dot{\varphi}_\varepsilon^{-1}(s))^2\, ds$$

$$\overset{\varphi_\varepsilon(t) = s}{=} \int_0^1 |\dot{\gamma}(t)|^2 \cdot |\dot{\varphi}_\varepsilon^{-1}(\varphi_\varepsilon(t))|^2\, dt$$

$\updownarrow$ equal

$$= \int_0^1 |\dot{\gamma}(t)|^2 \cdot \frac{1}{1 + \varepsilon\lambda}\, dt$$

Since γ is a minimizer, $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E(\gamma_\varepsilon)=0$ which is equivalent to

$$\int |\dot\gamma|^2 \dot\lambda = 0 \qquad \forall \lambda \in C_c^1([0,1])$$

$$\Rightarrow |\dot\gamma| = \text{constant.}$$

Remark. We can actually use

1) $\inf L(\gamma) \leq \sqrt{\inf E}$
2) $\inf E \doteq \min$ (inf is actually a min)
3) The minimizer of the energy has constant speed.

To show that : a) $\sqrt{\inf E} = \inf L$

b) $\exists$ minimizer for L and it has constant reparametrization.

Proof. (Sketch). Let $\varepsilon \in (0,1)$

$$(P_\varepsilon) \qquad \inf \{ \underbrace{(1-\varepsilon) L^2(\gamma) + \varepsilon E(\gamma)}_{\text{it is } W^{1,2} \text{ l.s.c. and } W^{1,2} \text{ coercive}} : \gamma(0)=P,\ \gamma(1)=Q \}$$

1) $\exists \min \gamma_\varepsilon$
2) $\gamma_\varepsilon \circ \varphi$, φ diffeo of $[0,1]$, $E(\varphi_\varepsilon) \leq E(\varphi_\varepsilon \circ \varphi) \Rightarrow \gamma_\varepsilon$ is constant reparametrization

Therefore,

$$\inf L^2(\gamma) \leq L^2(\gamma_\varepsilon) \overset{\text{since } \gamma_\varepsilon \text{ has constant spd reparametrization}}{=} (1-\varepsilon) L^2(\gamma_\varepsilon) + \varepsilon E(\gamma_\varepsilon)$$
$$\leq (1-\varepsilon) L^2(\gamma) + \varepsilon E(\gamma)$$
$$\leq E(\gamma) \qquad \forall \gamma$$

and thus

$$E(\gamma_\varepsilon) = L^2(\gamma_\varepsilon) \leq E(\gamma)$$

we have γ_ε is a minimizer of E. Additionally

$$E(\gamma_\varepsilon) = E(\gamma_{\varepsilon'}) \qquad \forall \varepsilon, \varepsilon'$$

$$\Rightarrow \inf E = E(\gamma_\varepsilon) = E(\gamma_{\varepsilon'}) = (1-\varepsilon') L^2(\gamma_{\varepsilon'}) + \varepsilon' E(\gamma_{\varepsilon'}) \leq (1-\varepsilon') L^2(\gamma) + \varepsilon' E(\gamma) \quad \forall \gamma$$

Taking $\varepsilon' \to 0$, $\inf E \leq L^2(\gamma)$ $\forall \gamma$ which implies $\inf E \leq \inf L^2$.

Calculus of Variations (Lecture 24) 26/05/2026

Continuing Proof of Douglas-Rado

Note: wedge product in $\mathbb{R}^3$ is the same as cross product.

$$\text{Area}(\Phi) = \int |\partial_x \Phi \wedge \partial_y \Phi| = \int |\partial_x \Phi| \cdot |\partial_y \Phi| \cdot |\sin(\widehat{\partial_x \Phi\, \partial_y \Phi})|$$

$$\leq \int |\partial_x \Phi| \cdot |\partial_y \Phi| \quad (= \text{ if } \partial_x \Phi \cdot \partial_y \Phi = 0)$$

$$\leq \frac{1}{2} \int |\partial_x \Phi|^2 + |\partial_y \Phi|^2 \quad (= \text{ if } |\partial_x \Phi| = |\partial_y \Phi|)$$

$$=: D(\Phi)$$

Definition. A regular map $\Phi : D \longrightarrow \mathbb{R}^3$ is said to be a conformal parametrization of $\Phi(D) = \Gamma'$ if:

1) Φ is a parametrization, i.e. $|\partial_x \Phi \wedge \partial_y \Phi| \neq 0$
2) $|\partial_x \Phi| = |\partial_y \Phi|$, $\partial_y \Phi \cdot \partial_x \Phi = 0$

Sometimes are called conformal coordinates.

The first fundamental form of a parametrization is

$$\underset{\substack{\text{or } g(x,y) \\ g \text{ metric}}}{I(x,y)} = \begin{pmatrix} \partial_x \Phi \cdot \partial_x \Phi & \partial_x \Phi \cdot \partial_y \Phi \\ \partial_y \Phi \cdot \partial_x \Phi & \partial_y \Phi \cdot \partial_y \Phi \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

for a conformal parametrization, $E = G$, $F = 0$ $\left(\Leftrightarrow g_{ij} = \lambda(x,y)\,\delta_{ij},\ \lambda = E = G \right)$

Definition. A map $\Phi : D \longrightarrow \mathbb{R}^3$ is said to be weakly conformal if $|\partial_x \Phi| = |\partial_y \Phi|$, $\partial_x \Phi \cdot \partial_y \Phi = 0$ (might be zero).

What we have shown is that $\text{Area}(\Phi) \leq D(\Phi)$ with equality iff Φ is weakly conformal.

Exercise. Let Φ be a smooth conformal parametrization, then

$$H = \text{Mean curvature} = \operatorname{tr}(\mathrm{II}^{nd} \text{ fundamental form}) = \frac{\Delta \Phi}{|\partial_x \Phi \wedge \partial_y \Phi|}$$

We will use the following theorem in our disposal, but proof is omitted

Theorem (Morrey). Let $\Phi \in W^{1,2}(D; \mathbb{R}^3) \cap C^0(\bar{D}; \mathbb{R}^2)$. Then $\forall \varepsilon > 0$, $\exists \varphi_\varepsilon : \bar{D} \longrightarrow \bar{D}$ homeomorphism, $\varphi_\varepsilon \in W^{1,2}(\bar{D}; \bar{D})$ such that

$$\Phi_\varepsilon = \Phi \circ \varphi_\varepsilon \in W^{1,2}(D; \mathbb{R}^3) \cap C^0(\bar{D}; \mathbb{R}^3)$$

$$D(\Phi_\varepsilon) \leq \text{Area}(\Phi_\varepsilon) + \varepsilon = \text{Area}(\Phi) + \varepsilon$$

Corollary. $\inf\limits_{\Phi \in \tilde{C}(\Gamma)} \text{Area}(\Phi) = \inf\limits_{X \in \tilde{C}(\Gamma)} D(X)$

and thus, to find a minimal surface would be enough to find a minimizer for

$$\inf_{\tilde{C}(\Gamma)} D(X)$$

We will not follow this path and we will directly prove the following

Theorem. Let $\Gamma \subset \mathbb{R}^3$ be a smooth Jordan curve, then

1) $\inf\limits_{\tilde{C}(\Gamma)} A(\Phi) = \inf\limits_{\tilde{C}(\Gamma)} D(\Phi)$

2) $\exists \tilde{\Phi} \in C^2(D; \mathbb{R}^3) \cap C^0(D; \mathbb{R}^3)$ which achieves the infimum
3) $\tilde{\Phi}|_{\partial D}$ is a homeomorphism between S^1 and Γ
4) $\tilde{\Phi}$ is weakly conformal

↳ Adapted from Chapter 4 of "Minimal Surfaces" Hildebrant, Dierkes, Sauvigny. Section 4.10

To prove the theorem we would like to minimize

$$(1-\varepsilon)\,\text{Area}(\Phi) + \varepsilon\, D(\Phi)$$

Proposition: Let $q \in D$, $\theta \in [0,1]$, define

$$\varphi_{q,\theta}(z) = e^{i\theta} \frac{z+q}{1+\bar{q}z} \quad \text{(Mobius Transformation)}$$

Then:

1) $\varphi_{q,\theta} : \bar{D} \longrightarrow \bar{D}$ is bijective
2) $\forall \Phi$, $D(\Phi) = D(\Phi \circ \varphi_{q,\theta})$

(More in general, if φ is conformal, $D\varphi = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, then $D(u \circ \varphi) = D(u)$)

Proof. It's only a computation, proof left as an exercise.

Remark. For $q_k \in D$, $q_k \longrightarrow +1$, then

$$\varphi_{q_k,0}(z) = \frac{z+q_k}{1+\bar{q}_k z} \xrightarrow[k\to\infty]{} \begin{cases} 1 & , z \in \bar{D} \setminus \{-1\} \\ ? & , z = -1 \end{cases}$$

The 3-Point-Condition

Let $P_i \in \Gamma$, $i \in \{1,2,3\}$ be three distinct points

$z_i \in \partial D = S^1$, $i \in \{1,2,3\}$ be three distinct points.

$$C(\Gamma) = \left\{ \Phi \in W^{1,2}(D;\mathbb{R}^3) \cap C^0(\partial D;\Gamma) : \Phi|_{\partial D} \text{ is weakly monotone parametrization of } \Gamma \text{ and } \Phi(z_i) = P_i \right\}$$

Lemma (Exercise).

Pick any $\Phi \in \overbrace{W^{1,2}(D;\mathbb{R}^3) \cap C^0(\partial D;\mathbb{R}^3) \cap \Phi_\Gamma \text{ weakly monotone parametrization}\}}^{\tilde{\tilde{C}}(\Gamma)}$

Then $\exists!\ \varphi_{q,\theta}$ and $\Phi \circ \varphi_{q,\theta} \in C(\Gamma)$ such that

$$\inf_{\tilde{\tilde{C}}(\Gamma)} A(x) = \inf_{C(\Gamma)} A(x) \quad \text{and} \quad \inf_{\tilde{\tilde{C}}(\Gamma)} D(x) = \inf_{C(\Gamma)} D$$

Lemma 1 (Coercivity of D on C(Γ)). Let $\Phi_k \in C(\Gamma)$ with $\sup_k D(\Phi_k) \leq C$.

Then up to subsequence, $\exists\ \Phi$ such that

1) $\Phi_k \xrightarrow{W^{1,2}} \Phi$
2) $\Phi_k|_{\partial D} \longrightarrow \Phi|_{\partial D}$ uniformly

$\longrightarrow \Phi \in C(\Gamma)$.

The Lemma is simple consequence of the following:

Lemma. Let Φ_k be as in Lemma 1. Then $\Phi_k|_{\partial D}$ is equicontinuous

Step 1 (Courant-Lebesgue Lemma). Let $\Phi \in W^{1,2}(D;\mathbb{R}^3)$

If $w \in \partial D$, $0<\delta<<1$. Then $\exists r \in (\delta, \sqrt{\delta})$ such that

$$\sup_{z_1,z_2\in \partial B_\delta(w)\cap D} |\Phi(z_1)-\Phi(z_2)| = \operatorname*{osc}_{\partial B_r(w)\cap D} \Phi \leq \frac{2\pi}{|\log \delta|}\left(\int_D |D\Phi|^2\right)^{1/2}.$$

Proof of Courant-Lebesgue:

By approximation, we can assume $\Phi\in C^1$, $r\in(\delta,\sqrt{\delta})$, $z_1,z_2\in\partial B_r(w)\cap D$,

$z_1 = w+re^{i\theta_1}$, $z_2 = w+re^{i\theta_2}$:

$$|\Phi(z_1)-\Phi(z_2)|^2 = \left|\int_{\theta_1}^{\theta_2} \frac{d}{d\theta}\Phi(z+re^{i\theta})\,d\theta\right|^2$$

$$\leq \int_{\theta_1}^{\theta_2} \left(|\nabla_\theta\Phi|(w+re^{i\theta})\right)^2 d\theta$$

$$\leq 2\pi\int_{\theta_1}^{\theta_2} |\nabla_\theta\Phi|^2(w+re^{i\theta})\,d\theta$$

$$\leq 2\pi r^2\int_0^{2\pi} |\nabla\Phi|^2(w+re^{i\theta})\,d\theta \quad \left(\begin{array}{c}\text{using change of variables}\\ |\nabla\Phi| = |\nabla_r\Phi| + \frac{1}{r^2}|D\Phi|\end{array}\right)$$

Let $G = \partial B_r(w)\cap D$. Therefore,

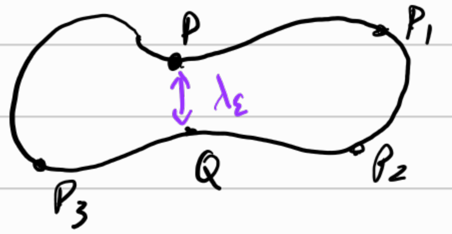
$$\frac{1}{r}\sup_{z_1,z_2\in G_r}|\Phi(z_1)-\Phi(z_2)|^2 \leq 2\pi r\int_0^{2\pi}|\nabla\Phi|^2(w+re^{i\theta})\,d\theta.$$

Integrate in $r\in(\delta,\sqrt{\delta})$:

$$\int_\delta^{\sqrt{\delta}} \sup_{z_1,z_2\in G}|\Phi(z_1)-\Phi(z_2)|^2\frac{dr}{r} \leq 2\pi\, D(\Phi)$$

$$\Rightarrow \underbrace{\left(\int_\delta^{\sqrt{\delta}}\frac{dr}{r}\right)}_{\frac{1}{2}|\log\delta|}\inf_{r\in(\delta,\sqrt{\delta})}\left(\operatorname*{osc}_{z_1,z_2\in G_r}\Phi\right)^2 \leq 2\pi\, D(\Phi).$$

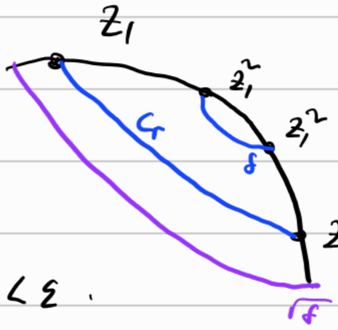
Step 2. $\forall \varepsilon>0$, $\exists \lambda_\varepsilon>0$ such that if $P, Q \in \Gamma$ with $|P-Q|<\lambda_\varepsilon$. Then $\operatorname{diam}(C_{PQ})<\varepsilon$, where C_{PQ} is the shortest arc in Γ between P, Q



Step 3. Fix $\varepsilon \ll 1$. In particular so small that $B_{10\cdot\varepsilon}(P_i)$ does not contain the others. Let λ_ε as in step 2, δ s.t.:

1) $B_{\sqrt{\delta}}(z_i)$ does not contain the other points
2) $\dfrac{4\pi}{|\log(\delta)|} C < \lambda_\varepsilon$

Let $\tilde{z}_1, \tilde{z}_2 \in \partial D$ such that $\operatorname{dist}(\tilde{z}_1, \tilde{z}_2)<\delta$.
We pick C_r as in step 1, $\{z_1, z_2\} = C_r \cap \partial D$.



By Step 1:

$$|\Phi(z_1)-\Phi(z_2)| \leq \frac{4\pi C}{|\log(\delta)|} < \lambda_\varepsilon \Rightarrow \operatorname{dist}\left(C_{\Phi(z_1)\Phi(z_2)}\right)<\varepsilon.$$

It can contain at most one P_i and also C_r contains at most one z_i.

By the 3-point-condition,

$$\Phi(\tilde{z}_1) \in \Phi\big(\partial D \cap \underbrace{\overset{\frown}{(z_1, z_2)}}_{\text{arc between } z_1,\, z_2}\big) \subseteq C_{\Phi(z_1)\Phi(z_2)}$$

Thus, $|\Phi(\tilde{z}_1)-\Phi(\tilde{z}_2)|<\varepsilon$.

By compactness and lower semicontinuity, $\exists\, \Phi^\varepsilon$ minimizer for $(1-\varepsilon)\operatorname{Area}(\Phi)+\varepsilon D(\Phi)$ on $\mathcal{C}(\Gamma)$

Lemma 3. Φ_ε is weakly conformal.

By Lemma 3 and argument from Lecture 23:

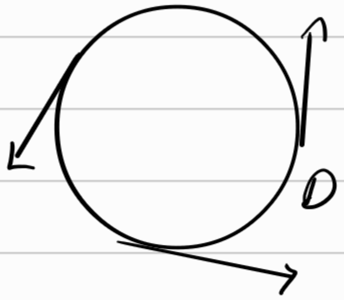
1) $\inf \text{Area} = \inf D$
2) Φ_ε minimizes Dirichlet energy $\Rightarrow \Phi_\varepsilon$ is harmonic $\Rightarrow C^2 \cap C^1$.

3) Φ_ε minimizes the area.

One can also prove $\Phi_\varepsilon|_{\partial D}$ is strictly monotone.

Proof of Lemma 3 (Sketch). Note that $D(\Phi_\varepsilon) = D(\Phi_\varepsilon \circ \varphi)$ for any $\varphi : \bar{D} \longrightarrow \bar{D}$. Let $\tau \in C^1(D; \mathbb{R}^2)$ such that $\tau(z) \perp \partial D$ on ∂D

we want a φ that solves the parabolic PDE.



$$\begin{cases} \partial_t \varphi(x) = \tau(\varphi) \\ \varphi(0,x) = 0 \end{cases}$$

$\Rightarrow$ φ_t of diffeomorphism from $\bar{D}$ into D

$$\varphi_t(x) = x + t\,\tau(x) + o(t)$$

$$\frac{d}{dt} D(\Phi_\varepsilon \cdot \varphi_t)\Big|_{t=0} = 0 \quad \text{by minimality}$$

where

$$\frac{d}{dt} D(\Phi_\varepsilon \circ \varphi_t)\Big|_{t=0} = \int_D (|\partial_x \Phi|^2 + |\partial_y \Phi|^2)(\tau_x^2 - \tau_y^2) + 2(\partial_x \Phi \cdot \partial_y \Phi)(\tau_x^2 + \tau_y^2)$$

Hopf differentials

$$\Rightarrow \quad 0 = \mathrm{Re}\left(\int_D (H(z)\, \partial_{\bar{z}} \tau)\right) \quad \text{where} \quad H(z) = |\partial_x \Phi|^2 - |\partial_y \Phi|^2 - 2i(\partial_x \Phi \cdot \partial_y \Phi)$$

$$\tau = \tau^1 + i\tau^2$$

$$\partial_{\bar{z}} = \partial_x + i\partial_y$$

Since $\tau \in C^1_c$, then $\partial_t H = 0$, H is holomorphic.

Since $\tau = -z\varphi$, $\mathrm{Im}(z^2 H(z)) = 0 \Rightarrow z^2 H(z) = \text{constant}$

$\Rightarrow H(z) = 0$

$\Rightarrow \Phi$ is conformal